ON THE SIMILARITY OF PLANE PULSED MAGNETIC FIELDS CONTINUED FROM DIFFERENT COORDINATE AXES

Purpose. The purpose of this work is formulation of similarity conditions for plane magnetic fields at a sharp skin-effect continued in non-conducting and non-magnetic medium from different axes bounding plane surfaces of conductors. Methodology. Classic formulation of Cauchy problem for magnetic vector potential Laplace equations, mathematic physics methods and basics similarity theory are used. Two problems are considered: the problem of initial field continuation from one axis and the problem of similar field continuation form other axis on which magnetic flux density or electrical field strength in unknown. Results. Necessary and sufficient similarity conditions of plane pulsed or high-frequency magnetic fields continued from different axes of rectangular coordinates are formulated. For the given odd and even magnetic flux density distributions on axis of initial field corresponding the distributions on axis and solution of continued similar field problem are obtained. Originality. It is proved that for similarity of examined fields the proportion of corresponding vector field projections represented by dimensionless numbers in similar points of axes is necessary and sufficient. References 11, figures 4.

Key words: plane magnetic field, sharp skin-effect, Cauchy problem for Laplace equation, similarity theory.

Introduction. The shape of massive solenoids (inductors) and electrodes used in electrophysical technologies to obtain electromagnetic fields of a given distribution is found by solving the field continuation problem [1-3]. We restrict ourselves to considering plane pulsed or high-frequency magnetic fields, continued from one of the axes of rectangular coordinates (for example, the x-axis) [4]. The problem definition includes the distribution of a certain projection of the vector of the extended field specified on this axis. In practice, it may be necessary to solve the problem of continuation of a field with a similar distribution on the y-axis. In this case, it is obvious to use the results obtained for the x-axis. The main difficulty of this approach lies, first of all, in insufficient theoretical substantiation, as a result of which the given field distribution on the y-axis turns out to be unknown.

The goal of the work is a formulation of conditions for the similarity of plane magnetic fields at a sharp skin-effect, which continue into a non-conductive and non-magnetic medium from different axes of rectangular coordinates that limit the flat surfaces of the conductors.

Conditions for the similarity of magnetic fields extended from flat surfaces of conductors. In a massive conductor with a flat boundary surface eddy currents are induced under the action of a pulsed or high-frequency magnetic field of an external inductor, the profile of...
which must be determined. The skin-effect is sharply manifested in the conductor. Let us accept the assumption of an ideal surface effect [3] and replace the conductor with an ideally superconducting half-space. We use three systems of Cartesian coordinates on the plane: the main (general) xOy and two auxiliary ones – x1Oy1 and x2Oy2.

Consider two corresponding problems of continuation of a plane magnetic field into non-magnetic non-conducting half-spaces \( y_1 > 0 \) and \( x_2 > 0 \) without sources (Fig. 1): from the \( x_1 \) axis (the first problem whose solution is known) and from the \( y_2 \) axis (the second problem). Half-spaces \( y_1 < 0 \) and \( x_2 < 0 \) are ideal superconductors.

Equation for the magnetic vector potential \( A(x, y) \) of such fields has the form [5]

\[
\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = 0 .
\]

(1)

In (1) \( x = x_1 \lor x_2, \quad y = y_1 \lor y_2, \quad A(x, y) = A(x_1, y_1) \lor A(x_2, y_2) \). Boundary conditions on the \( x_1 \)-axis –

\[
A_1(x_1, 0) = 0, \quad \frac{\partial A_1}{\partial y_1}
\bigg|_{y_1=0} = B_{1x}(x_1, 0),
\]

(2, 3)

on the \( y_2 \)-axis –

\[
A_2(0, y_2) = 0, \quad \frac{\partial A_2}{\partial y_2}
\bigg|_{y_2=0} = -B_{2y}(0, y_2),
\]

(4, 5)

where \( B_{1x}(x, 0) \) and \( B_{2y}(0, y) \) are the projections of the magnetic flux density \( B_1(x_1, y_1) \), \( y_1 \geq 0 \) and \( B_2(x_2, y_2) \), \( x_2 \geq 0 \) on the \( x_1, y_2 \) axes.

Let us assume that the considered magnetic fields are similar. Then it follows from (1) – (5) that the similar coordinates \([6-8]\) in the field continuation domains are \( x_1 \) and \( y_2 \), \( y_1 \) and \( x_2 \), and the corresponding functions are \( A_1(x_1, y_1) \) and \( A_2(x_2, y_2) \). Therefore, analogous quantities characterizing supposedly similar fields should be

\[
\frac{\partial A_1}{\partial x_1} \quad \frac{\partial A_2}{\partial y_2}, \quad \frac{\partial A_1}{\partial y_1} \quad \frac{\partial A_2}{\partial x_2}, \quad or
\]

\(-B_{1x}(x_1, y_1) \quad B_{1x}(x_2, y_2), \quad B_{2y}(x_1, y_1) \quad -B_{2y}(x_2, y_2).
\)

Let \( P \) be the observation point of the field with coordinates \( x_P = x_{1P} \lor x_{2P}, y_P = y_{1P} \lor y_{2P} \) (Fig. 1). Then the similar coordinates of the point \( P \) will be \( x_{1P} \lor y_{2P}, y_{1P} \lor x_{2P}, y_P \).

Taking into account the main provisions of the similarity theory [6-8], in addition to the noted necessary conditions, we can assert the following: for the similarity of two compared magnetic fields, it is necessary and sufficient that the values of the presented in criterial form magnetic flux density projections \( B_{1x}(x_1, 0) \) and \( B_{2y}(0, y_2) \) at similar points of the axes, from which these fields continue, should be proportional.

This condition allows to find \( B_{2y}(0, y_2) \) and thus obtain a complete formulation of the second problem.

Magnetic flux density distribution on the \( y_2 \) axis for a similar magnetic field. Let in the first problem the given distributions of the magnetic flux density on the \( x_1 \)-axis can be represented by the formula

\[
B_{1x}(x_1, 0) = \frac{\mu_0 I_{1m}}{\pi} \left\{ \frac{1}{(x_1 - x_{1m})^2 + y_{1m}^2} \right\}.
\]

(8)

where \( \mu_0 \) is the magnetic constant, \( I_{1m}, x_{1m}, y_{1m} \) are the distribution parameters.

The function in curly brackets of formula (8) is odd or even, depending on the minus or plus sign between the terms in square brackets. In both cases, it has the well-known sine or cosine Fourier transform. The multiplier before of the considered function is constant therefore \( B_{1x}(x_1, 0) \) also has such transform. The physical meaning of (8) is the magnetic flux density created on the \( x_1 \)-axis by a system of four parallel, symmetrically located axes with currents \( \pm I_{1m} = \pm I_{1m}, \) two of which \( (M_1', M_2') \) replace the influence of the lower ideally superconducting half-space \([9, 10]\) (Fig. 1). The parameters \( x_{1m}, y_{1m} = x_{1m}, y_{1m} \) determine the position of the axes at the points \( M_1, M_1', M_2, M_2' \) of the \( x_1Oy_1 \) plane. Currents \( \pm I_{1m} \) have positive directions, and \( -I_{1m} \) have negative ones, indicated by a dot or a cross, respectively. For the currents in Fig. 1 we obtain an odd magnetic flux density distribution. If the currents in the upper half-space have the same (for example, positive) direction (while the currents in the lower half-space are also directed in the same way, but opposite to the first ones), we have an even distribution of magnetic flux density.

Let us find such distributions \( B_{2y}(0, y_2) \) on the \( y_2 \)-axis that satisfy the necessary and sufficient similarity condition formulated above. To do this, we first establish
similar parameters [6-8] of the distributions \(B_1(x_1, 0)\) and \(B_2(0, y_2)\): \(I_{1M}\) and \(I_{2M}\), \(x_{1M}\) and \(y_{1M}\), \(x_{2M}\) and \(y_{2M}\), where \(I_{1M}\), \(x_{2M}\) and \(y_{2M}\) are the parameters of the distributions \(B_2(0, y_2)\) unknown so far. Then, using the correspondence of similar values (7), we replace in formula (8) the coordinates and parameters with similar coordinates and parameters of the second problem. We obtain:

\[
B_{2y}(0, y_2^*) = -\frac{\mu_0 I_{2y}^{*}}{\pi} \left[ x_{2y}^{*} \left( \frac{1}{(y_2^* - y_2^{*2})^2 + x_{2y}^{*2}} \right) \right] \tag{9}
\]

We represent (8), (9) in dimensionless form (in criterial form) using two systems of basic quantities: \(1_{b}\) and \(I_{2b}\) – length, \(I_{1b}\) and \(I_{2b}\) – current, \(B_{1b}\) and \(B_{2b}\) – magnetic flux density (basic values for formula (8) have number 1 in the subscript, for (9) – number 2). Dimensionless quantities are obtained by dividing the corresponding dimensional ones by the basic ones and marked with asterisks. After transformations formulas (8), (9) take the following form:

\[
B_{1x}^*(x_1^*, 0) = \frac{1}{\pi} \frac{I_{1y}^{*}}{x_{1y}^{*}} \left[ \frac{1}{(x_1^{*} - x_{1y}^{*}) + y_{1y}^{*2}} \right] \tag{10}
\]

\[
B_{2x}^*(0, y_2^*) = \frac{1}{\pi} \frac{I_{2y}^{*}}{x_{2y}^{*}} \left[ \frac{1}{(y_2^{*} - y_{2y}^{*2}) + x_{2y}^{*2}} \right] \tag{11}
\]

Comparing (10) and (11), we see that for

\[
x_{1y}^{*} = y_{1y}^{*}, \quad y_{1y}^{*} = x_{2y}^{*}
\]

at similar points on the \(x_1\) and \(y_2\) axes with coordinates \(x_1^* = y_2^*\), the values \(B_{1x}^*(x_1^*, 0)\) are proportional to the values \(B_{2x}^*(0, y_2^*)\). Consequently, the necessary and sufficient similarity condition is satisfied, and the sought distributions \(B_2(0, y_2)\) for such a magnetic field have the form (9). If in addition to (12) to accept

\[
I_{1y}^{*} = I_{2y}^{*},
\]

then the absolute values of the compared magnetic flux density values will be equal, although this is not necessary for similarity.

All quantities included in conditions (12), (13) are similarity criteria. We choose the basic values \(I_{1b}\), \(I_{2b}\) in such a way that conditions (12) are satisfied. In the general case, \(I_{1b}\) and \(I_{2b}\) can be any, but, if necessary, we find them taking into account condition (13). When determined \(B_{1x}^*(x_1^*, 0)\) and \(B_{2x}^*(0, y_2^*)\), we accepted \(B_{1y} = \mu_0 I_{1y}/I_{1b}, \quad B_{2y} = \mu_0 I_{2y}/I_{2b}\).

The physical meaning of distributions (9) is similar to that described for (8): the magnetic flux density created on the \(y_2\) axis by four parallel axes with currents \(\pm I_{1y} = \pm I_{2y}\) (the axes are located at points whose coordinates \(\pm x_{1y}\) and \(\pm y_{1y}\) are determined by the parameters \(\pm x_{2y}\) and \(\pm y_{2y}\), Fig. 1).

Figure 2 shows the symmetric parts of the odd (a) and even (b) distributions of the magnetic flux density on the axes at \(x_1 \geq 0\) and \(y_2 \geq 0\), calculated by (10), (11). Accepted: \(I_{1b} = I_{2b}, \quad I_{1b} = I_{2b}, \quad I_{1b} = I_{2b}\) and curves 1 – \(x_{1y} = 0.1, \quad y_{1x} = 0.1; 2 – 0.25, 0.1; 3 – 0.15, 0.2; 4 – 0.25, 0.2\). The values \(x_{2y}^*\) and \(y_{2y}^*\) are determined using relations (12). The coincidence of the distributions \(B_{1x}^*(x_1^*, 0)\) and \(B_{2x}^*(0, y_2^*)\) illustrates the necessary and sufficient condition for the similarity of magnetic fields.

Fig. 2. Odd (a) and even (b) distributions of the magnetic flux density on the axes \(x_1, y_2\)

Continuation of similar magnetic fields by solving the first problem. The solutions of the first problem (1)-(3), taking into account (8), obtained by the method of particular solutions that continuously depend on the parameter, have the following form:
By using the known relationship between the magnetic flux density and the vector potential of the magnetic field [5] and (16) to calculate the projections, we obtain the following formulas:

\[ B_{1x}(P) = -\frac{\mu_0 I_M}{2\pi} \left( y_P - y_M \left( \frac{1}{r_{M,P}^2} - \frac{1}{r_{M,P}^2} \right) \right) + \left( y_P + y_M \right) \left( \frac{1}{r_{M,P}^2} - \frac{1}{r_{M,P}^2} \right) \]  
\[ B_{1y}(P) = \frac{\mu_0 I_M}{2\pi} \left( x_P - x_M \left( \frac{1}{r_{M,P}^2} - \frac{1}{r_{M,P}^2} \right) \right) + \left( x_P + x_M \right) \left( \frac{1}{r_{M,P}^2} - \frac{1}{r_{M,P}^2} \right) \]  

(17) \hspace{1cm} (18)

Note that in (16) – (18) it is assumed that the observation point \( P \) is located in the upper half-space \( y > 0 \) (in a particular case on the \( x \)-axis). Let us find \( B_{2x}(P) \) and \( B_{2y}(P) \) for a similar magnetic field in the region \( x > 0 \) (in a particular case on the \( y \)-axis), replacing coordinates and parameters in (17), (18) with similar values. We obtain surprising, at first glance, results: the formulas for determining \( B_{2x}(P) \) and \( B_{2y}(P) \) formally coincide with (17), (18). The reason is that a system of four axes with currents, which creates a magnetic field in the region \( y > 0 \) of the first problem (for more details, in the physical sense of (8)), simultaneously creates a similar magnetic field in the region \( x > 0 \). Here, the axes located in points \( M_2 \) and \( M_2' \) (Fig. 1), replace the influence of an ideally superconducting half-space \( x < 0 \). Therefore, formula (16) is also a solution of the second problem for a similar magnetic field in the region \( x > 0 \) in the case of an odd distribution \( B_3(0, y_2) \).

When using the Green function in the case of even distributions \( B_1(0, y) \) and \( B_2(0, y_2) \), it is necessary to change the directions of currents in two axes to the opposite with respect to those adopted in Fig. 1: for the original field – in the \( M_1 \) and \( M_1' \) axes, for a similar field – in the \( M_2 \) and \( M_2' \) axes. In contrast to odd distributions of the magnetic flux density on the axes, the vector potential \( A(P) \) is described by two different formulas. We obtain them from formula (16), having changed places \( r_{M_1,P} \) and \( r_{M_2,P} \) for the original field, \( r_{M_1,P} \) and \( r_{M_2,P} \) for a similar field:

\[ A_1(P) = \frac{\mu_0 I_M}{\pi} \ln \left[ \frac{r_{M_1,P}^2 + r_{M_1,P}^2}{r_{M_1,P}^2 + r_{M_2,P}^2} \right] \]  
\[ A_2(P) = \frac{\mu_0 I_M}{\pi} \ln \left[ \frac{r_{M_1,P}^2 + r_{M_2,P}^2}{r_{M_1,P}^2 + r_{M_2,P}^2} \right] \]  

(19) \hspace{1cm} (20)

Formulas for calculating magnetic flux density projections \( B_{1x}(P) \) and \( B_{1y}(P) \), \( B_{2x}(P) \) and \( B_{2y}(P) \) differ from (17), (18) in opposite signs before the fractions \( 1/r_{M_1,P}^2 \) and \( 1/r_{M_2,P}^2 \) which confirms the correctness of the obtained results to the relations (6), (7).

Figures 3, 4 show the magnetic field lines of the initial and similar magnetic fields \( A(x, y) = \text{const} \), calculated by (16), (19), (20) for the distributions of the magnetic flux density \( 2 \) in Fig. 2.a,b. It is accepted that \( A = A/\mu_0, A_0 = \mu_0 I_M, A_1 = A_{10} \vee A_{20}, A_1' = A_{10}', A_{20}' = A_{20}, l_{10} = l_{20} \).

For magnetic field lines 1, 5 – \( A_1 = A_2 = 0.05, 2, 6 - 0.1, 3, 7 - 0.15, 4, 8 - 0.2 \).

We see that the corresponding field lines of the considered magnetic fields are geometrically similar, which confirms the correctness of the obtained results. The field lines shown in Fig. 4,a,b, limit the profiles of current-conducting inductors to create pulsed or high-frequency magnetic fields of given distributions on the axis \( y_2 \).
The results obtained for the magnetic field can be used to determine the profiles of one or more long parallel uniformly charged electrodes, with the help of which an electrostatic field of a given distribution is to be created on the flat surface of the conductor. For this we use the electrostatic analogy of plane electrostatic and magnetic fields of conductors with a sharp skin-effect (J.D. Cockroft, 1929, [4]), according to which the distributions of the taken with a minus sign projection of the electric field strength $E_i(x, 0)$ and $B_i(x, 0)$ correspond to one another.

**Appendix.** The use of two methods for solving field continuation problems allows not only checking the results, but also obtaining formulas for calculating complex improper integrals that are absent in the reference literature [11]. For example, comparing formulas (14) and (19) for the initial field, we have two improper integrals:

$$
\int_0^{\infty} e^{-y M^2 \lambda^2} \left\{ \frac{\sin(y M \lambda) \sin(x \lambda)}{\cos(x M \lambda) \cos(y \lambda)} \right\} \sin(y \lambda) d\lambda =
$$

$$
= \frac{1}{8} \ln \left\{ \frac{(x-x_M)^2 + (y+y_M)^2}{(x-x_M)^2 + (y-y_M)^2} \right\} \times
$$

$$
\left[ \frac{(x-x_M)^2 + (y+y_M)^2}{(x+x_M)^2 + (y+y_M)^2} \right] \times
$$

$$
\left[ \frac{(x+x_M)^2 + (y+y_M)^2}{(x-x_M)^2 + (y-y_M)^2} \right]
$$

$$
\int_0^{\infty} e^{-y M^2 \lambda^2} \left\{ \frac{\sin(y M \lambda) \sin(x \lambda)}{\cos(x M \lambda) \cos(y \lambda)} \right\} \cos(x \lambda) d\lambda =
$$

$$
= \frac{1}{8} \ln \left\{ \frac{(x-x_M)^2 + (y+y_M)^2}{(x-x_M)^2 + (y-y_M)^2} \right\} \times
$$

$$
\left[ \frac{(x+x_M)^2 + (y+y_M)^2}{(x-x_M)^2 + (y+y_M)^2} \right] \times
$$

$$
\left[ \frac{(x+x_M)^2 + (y+y_M)^2}{(x-x_M)^2 + (y-y_M)^2} \right]
$$

The limits of $x$ and $y$ change are the same as in (14). In the described way, one can obtain several more formulas for calculating improper integrals using (17), (18), as well as the corresponding formulas for a similar field. Comparison of (15) with (20) leads to a number of improper integrals. For instance:

$$
\int_0^{\infty} e^{-y M^2 \lambda^2} \left\{ \frac{\sin(y M \lambda) \sin(y \lambda)}{\cos(y M \lambda) \cos(y \lambda)} \right\} \sin(x \lambda) d\lambda =
$$

$$
= \frac{1}{8} \ln \left\{ \frac{(x+x_M)^2 + (y-y_M)^2}{(x-x_M)^2 + (y-y_M)^2} \right\} \times
$$

$$
\left[ \frac{(x+x_M)^2 + (y+y_M)^2}{(x-x_M)^2 + (y+y_M)^2} \right] \times
$$

$$
\left[ \frac{(x+x_M)^2 + (y+y_M)^2}{(x-x_M)^2 + (y-y_M)^2} \right]
$$
We see that formula (22) differ from (21) only by similar values of a similar magnetic field. In addition, it is necessary to take into account also other limits of variation of x and y (see formula (15)).

**Conclusions.**

1. For the similarity of plane pulsed or high-frequency magnetic fields continued into a non-magnetic and non-conductive medium from different axes of Cartesian coordinates that bound the flat surfaces of the conductors, it is necessary and sufficient that the values of the corresponding projections of the magnetic flux density presented in the criterial form at similar points of the axes are proportional. This condition makes it possible to find the distribution of the magnetic flux density on the axis from which the similar field continues.

2. Solutions to the problems of the continuation of similar magnetic fields can be obtained from the known solutions of the problems of the continuation of the initial fields by replacing the coordinates and parameters in them with the corresponding similar quantities.

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