THE RECIPROCY PRINCIPLE FOR A NONLINEAR ANISOTROPIC MEDIUM WITHOUT HYSTERESIS: THEORY AND PRACTICE OF APPLICATION

The construction of the correct vector material equations for nonlinear anisotropic soft magnetic materials remains one of the main reserves for increasing the accuracy of mathematical models in solving magnetostatic problems in the field formulation. The aim of the work is to establish asymptotic expressions for the reciprocity principle, which is a fundamental property of reversible magnetization processes of nonlinear anisotropic media, and to use the obtained results to optimize the computational process when constructing the vector magnetization characteristic and differential permeability tensor. The potentiality property of the magnetic flux density vector $B$ in $H$-space is used. The main result of the paper is an illustration, using concrete examples, of an alternative method for calculating vector magnetization characteristics for one of the orthogonal families. In order to eliminate the instrumental error and ensure maximum accuracy and reliability of the obtained results, the exact characteristics for the components of the vector magnetization characteristic obtained by differentiating a special analytical expression for the potential were used as initial ones. The principle of reciprocity, by virtue of its universal nature, makes a significant contribution for the components of the vector magnetization characteristic obtained by differentiating a special analytical expression for the electrical steels, which are most used in electrical engineering. Asymptotic expressions, magnetic permeability tensor. Key words: nonlinear anisotropic medium, vector magnetization characteristics, energy potential, reciprocity principle, asymptotic expressions, magnetic permeability tensor.
relation (1) formally coincides with the classical condition of potentiality, since an expression under the sign of the integral is the total differential of the potential \( \Psi(H) \).

The above definition of \( H \)-space indicates the independence of the potential \( \Psi(H) \) from the integration path in \( H \)-space, the eddy-free character of the magnetic flux density field \( B \), which is the force vector of the field in this space:

\[
B = \frac{d\psi}{dH} = \text{grad}_H \Psi(H) = i_1 B_1(H_1, H_2) + i_2 B_2(H_1, H_2).
\]  

(2)

In addition, we note an important consequence of relations (1) and (2) – the symmetry of the differential magnetic permeability tensor \( \mu_d(H) \):

\[
\mu_d = \frac{dB}{dH} ; \quad \mu_{dij} = \frac{\partial^2 \psi}{\partial H_i \partial H_j} = \frac{\partial^2 \psi}{\partial H_j \partial H_i} = \mu_{dij}.
\]  

(3)

In formulas (2), (3), we use the notation associated with the concepts of the derivative of a scalar and a vector function with respect to the vector argument used in vector algebra [12, items 6.2, 6.3].

Energy potentials cannot be measured directly; therefore, their construction is a difficult problem even in the two-dimensional case [7]. The basic information for construction of the potential is a certain set of computational process when constructing the vector characteristics of anisotropic materials, we note the obvious problems with the numerical differentiation of the two-dimensional case [7]. The basic information for construction of the potential is the total differential of the potential \( \Psi(H) \) from partially given information about the magnetic properties of a nonlinear anisotropic medium.

In practical terms, the importance of the reciprocity principle lies in the possibility of constructing a vector magnetization characteristic \( B(H) \) from partially given information about the magnetic properties of a nonlinear anisotropic medium.

**The goal** of this paper is further generalization of the reciprocity principle (4), in particular, obtaining its asymptotic expressions and using them to optimize the computational process when constructing the vector characteristic \( B(H) \) and the differential magnetic permeability tensor \( \mu_d(H) \). As we know, it is this information about the magnetic properties of the medium that is used in various computational schemes.

### Asymptotic expressions for the principle of reciprocity.

**A. The case of Cartesian coordinates.** We write relations (4) in relation to Fig. 1:

\[
S_1 = \int \frac{[B_1(H_1, H_2^\ast + \delta) - B_1(H_1, H_2^\ast - \delta)]}{H_1^\ast + \delta} dH_1^\ast;
\]  

\[
S_2 = \int \frac{[B_2(0, H_2) - B_2(H_1^\ast, H_2^\ast)]}{H_2^\ast - \delta} dH_2^\ast.
\]  

(5)

Since relations (5) are valid for arbitrary values of \( \delta \), we consider the limiting expressions for the integrals \( R_1^* = R_2^* \):

\[
R_1^* = \lim_{\delta \rightarrow 0} \int_0^{H_1^\ast} \frac{[B_1(H_1, H_2^\ast + \delta) - B_1(H_1, H_2^\ast - \delta)]}{H_1^\ast + \delta} dH_1^\ast;
\]  

(6)

\[
R_2^* = \lim_{\delta \rightarrow 0} \int_{H_2^\ast - \delta}^{H_2^\ast} \frac{[B_2(0, H_2) - B_2(H_1^\ast, H_2^\ast)]}{H_2^\ast - \delta} dH_2^\ast.
\]

For small values of \( \delta \) the integrand in the first integral (6) can be expressed in terms of differential magnetic permeability \( \mu_{d12} \):

\[
\lim_{\delta \rightarrow 0} [\Delta B_1(H_1, H_2^\ast) / \Delta H_1^\ast] \approx \mu_{d12}(H_1, H_2^\ast),
\]

therefore, the integral \( R_1^* \) becomes curved and takes the form

\[
R_1^* = \int_0^{H_1^\ast} \frac{\mu_{d12}(H_1, H_2^\ast)}{H_1^\ast} dH_1^\ast.
\]  

(7)

As for the integral \( R_2^* \), as can be seen from Fig. 1, at \( \delta \rightarrow 0 \), the area \( S_2 \) degenerates into a line \( ab \), which corresponds to the increment of the magnetic flux density component \( B_2 \) at \( H_2^* = H_2^\ast \) and \( 0 \leq H_1 \leq H_1^\ast \). Therefore

![Fig. 1. Geometric meaning of the principle of reciprocity](Image 308x691 to 420x785)
the restoration of potential respectively (4) and (7), (8). Thus, the reciprocity principle has three possible representations: point one (at each «point» reciprocity principle has three possible representations: established previously by relation (3). Thus, the potential from a given family of characteristics of «transverse» magnetization orthogonal to the orthogonal characteristics \( B_\| (H, \alpha) \) without calculating the potential \( \Psi \) using expression (8).

Note that by rearranging the indices, we can obtain relations similar to (7), (8) for other initial data, for example, \( B_\| (H, 0) \) and \( B_\perp (H, 0) \):

\[
R^*_2 = \int_{H_2^*}^H \mu_{d12}(H_2, H_2^*) \cdot dH_2; \quad (9)
\]

\[
R^*_1 \rightarrow cd = \int_{H_2^*}^H \mu_{d12}(H_2, H_1^*) \cdot dH_2, \quad (10)
\]

and \( R^*_1 = R^*_2 \).

B. The case of polar coordinates. Relations (1), (2) are invariant with respect to the coordinate system. We assume that the family of «longitudinal» magnetization characteristics \( B_\parallel (H, \alpha) \), where \( B_\| \) is the projection of the vector \( B \) onto the vector \( H \) and \( \alpha \) is the angle defining the direction of the vector \( H \) is specified (basic) information. For the vector \( B \), we use the decomposition \( B = B_\parallel (H, \alpha) + B_\perp (H, \alpha) \), where \( B_\| (H, \alpha) \) is the family of characteristics of «transverse» magnetization orthogonal to \( B_\parallel (H, \alpha) \).

In view of the above, formula (2) takes the form

\[
B = \frac{d\Psi}{dH} = \text{grad}_H \Psi(H) = r^0 \frac{\partial \Psi}{\partial H} + \alpha^0 \frac{1}{H} \frac{\partial \Psi}{\partial \alpha}, \quad (11)
\]

where \( r^0, \alpha^0 \) are the unit vectors of the polar coordinate system.

We note the possibility of reconstructing the potential from a given family of characteristics \( B_\parallel (H, \alpha) \). So, taking \( \Psi(0) = 0 \), for an arbitrary point \( H = (H, \alpha) \) we obtain

\[
\Psi(H) = \int_0^H B_\parallel(H, \alpha) \cdot dH
\]

and \( B_\perp (H, \alpha) = \frac{1}{H} \frac{\partial \Psi}{\partial \alpha} \).

As in the case of Cartesian coordinates, to calculate the orthogonal characteristics \( B_\perp (H, \alpha) \) without calculating the potential, we use the reciprocity principle.

Let \( H' \leq H \leq H' \) and \( \alpha' \leq \alpha \leq \alpha'' \). We have the equality of integrals \( S_1 \) and \( S_2 \):

\[
S_1 = \int_0^H [B_\parallel(H, \alpha') - B_\parallel(H, \alpha'')] \cdot dH; \quad (12)
\]

\[
S_2 = \int_0^H [H' \cdot B_\perp(H', \alpha) - H'' \cdot B_\perp(H'', \alpha)] \cdot d\alpha.
\]

The proof and illustration of relations (12) are given in [9]. We establish the asymptotic properties of the reduced integral reciprocity principle, similar to relations (9) and (10). Let \( H' = (H', \alpha') \) be given – some point in the intervals \( H' \leq H' \leq \alpha' \leq \alpha'^* \leq \alpha'' \) and \( \delta \alpha \) – the deviation of the angle \( \alpha \) from this point. Then, by analogy with (5) for \( H' = 0 \) and \( H' = H' \) (Fig. 2)

\[
S_1^* = \int_0^H [B_\parallel(H, \alpha^* + \delta \alpha) - B_\parallel(H, \alpha^* - \delta \alpha)] \cdot dH; \quad (13)
\]

\[
S_2^* = \int_0^H -H' \cdot B_\perp(H', \alpha) \cdot d\alpha.
\]

Limit expressions for these relations

\[
S_1^* = \lim_{\delta \alpha \to 0} \int_0^H [B_\parallel(H, \alpha^* + \delta \alpha) - B_\parallel(H, \alpha^* - \delta \alpha)] \cdot dH; \quad (14)
\]

\[
S_2^* = \lim_{\delta \alpha \to 0} \int_0^H -H' \cdot B_\perp(H', \alpha) \cdot d\alpha.
\]

The first integral in (14) can be expressed in terms of the differential magnetic permeability \( \mu_{dH/\alpha} \). The integral \( R^*_2 \), as can be seen from Fig. 2, for \( \delta \alpha \to 0 \), by analogy with (7), (8), degenerates into the line \( cd \):

\[
R^*_1 \rightarrow cd = \int_0^H \mu_{dH \alpha}(H, \alpha^*) \cdot dH. \quad (15)
\]

Since for \( H = 0 \) \( H \cdot B_\perp = 0 \) for all \( \alpha \), the value of the segment \( cd \) determines the value of \( H' \cdot B_\perp(H') \) and, therefore, \( B_\perp(H') \). Similarly, we can calculate the remaining components of the array \( B_\perp(H) \).

**Fig. 2. Geometric meaning of the principle of reciprocity for polar coordinates**

**Computational experiments and discussion.** In order to eliminate the instrumental error and ensure maximum accuracy and reliability of the results, we use the exact magnetization characteristics \( B_\parallel(H_1, H_2) \) and...
$B_2(H_1, H_2)$ obtained by differentiating the potential given in [8] (Fig. 3.a,b).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{Nonlinear anisotropic medium magnetization characteristics obtained by differentiation of the potential [8]}
\end{figure}

It follows from the reciprocity principle that the family of characteristics $B_2(H_1, H_2)$ can be calculated from the characteristics $B_1(H_1, H_2)$ and one of the characteristics $B_2(H_1, H_2)$, for example, $B_2(0, H_2)$. This information is reflected in Fig. 3.a,b by the symbols «••» for nodes in which the values of magnetic flux density will be considered known. The symbols «••» correspond to the calculated values obtained by using expression (8) for the missing mesh nodes. From Fig. 3.b, one can see almost complete coincidence with the calculated dependences $B_2(H)$ (solid lines).

The algorithm of «filling» information on the magnetic properties of a nonlinear anisotropic medium is as follows. Using the known array of characteristics $B_1(H)$, by differentiation we obtain three components of the tensor $\mu_2(H)$: $\mu_{d11}(H) = \partial B_1(H) / \partial H_1$; $\mu_{d12}(H) = \partial B_1(H) / \partial H_2 = \mu_{d21}(H)$. Then, using expression (8), we find the corresponding values of the integrals $ab$, the subtraction of which from the values of the given characteristic $B_2(0, H_2)$ determines the family of characteristics $B_2(H)$ and, finally, $\mu_{d22}(H) = \partial B_2(H) / \partial H_2$.

To confirm, we give some numerical examples. We choose two arbitrary vectors of magnetic field strength $H$, for example, $H^{(1)} = (450, 600) \text{ A/m}$ and $H^{(2)} = (1120, 375) \text{ A/m}$. The calculated values of the corresponding magnetic flux density vectors $B^{(1)} = (1.259, 1.005) \text{ T}$ and $B^{(2)} = (1.669, 0.275) \text{ T}$. And here, if the values of the $B_1$ components (1.259 T and 1.669 T, respectively) are obtained by spline interpolation of a given array of nodal values of the magnetic flux density $B_1(H_1, H_2)$ (see Fig. 3.a), then the corresponding values of the $B_2$ components are calculated by the above technique without calculating the potential $\Psi$. The exact values of the magnetic flux density vectors obtained by using the analytical expressions from [8]: $B^{(1)} = (1.259, 1.022) \text{ T}$, $B^{(2)} = (1.670, 0.257) \text{ T}$. The mismatch angles between the vectors $B$ and $H$ are respectively equal 9.75° and 14.07°.

We also give the calculated and exact ($\ast$) values of the differential absolute magnetic permeability tensors for given values $H^{(1)}$ and $H^{(2)}$:

\begin{align*}
\mu^{(1)}_d &= \begin{bmatrix} \mu_{d11}^{(1)} & \mu_{d12}^{(1)} \\ \mu_{d21}^{(1)} & \mu_{d22}^{(1)} \end{bmatrix} = 10^{-3} \begin{bmatrix} 1.28247 & -1.25014 \\ -1.25014 & 1.54950 \end{bmatrix} \\
\mu^{(1)*}_d &= 10^{-3} \begin{bmatrix} 1.26986 & -1.23883 \\ -1.23883 & 1.56906 \end{bmatrix}
\end{align*}

\begin{align*}
\mu^{(2)}_d &= 10^{-4} \begin{bmatrix} 0.53929 & -2.19504 \\ -2.19504 & 10.7357 \end{bmatrix} \\
\mu^{(2)*}_d &= 10^{-4} \begin{bmatrix} 0.53756 & -2.21087 \\ -2.21087 & 11.2446 \end{bmatrix}
\end{align*}

We note that the given values of the tensors $\mu_2(H)$ are local, therefore, from the fact that for the selected values of the field strength vectors $\mu_{d22} > \mu_{d11}$, it cannot be concluded that the $H_2$ axis is the direction of easy magnetization. As will be illustrated in Fig. 4, this direction is the axis $H_1$.

We also note one of the useful consequences of the integral reciprocity principle: at the same scales for the corresponding components of the vectors $B$ and $H$, the areas bounded by the limiting magnetization curves are the same, according to (4) the areas of all the corresponding curved quadrangles are the same. For shown in Fig. 3.a,b boundary characteristics $B_1(H_1, 0)$ and $B_2(H_1, 1200)$, $B_2(0, H_2)$ and $B_2(1200, H_2)$, by integration almost identical values were obtained: 880.4670 J and 880.4688 J. This property can be useful in conditions of limited information on the magnetic properties of an anisotropic medium, when only two characteristics are specified in orthogonal directions.

To further illustrate the anisotropic properties of the considered medium, Fig. 4 shows the hodographs of the vector $H$ (semicircles of radii of 1200, 600, and 300 A/m) and the corresponding hodographs of the magnetic flux density vector $B$. The anisotropy of the medium is manifested by a pronounced nonlinear dependence of magnetic flux density on the field strength, more «easy» magnetization in the direction of the $H_1$ axis, and significant mismatch between vectors $B$ and $H$ in almost the entire range of field changes.

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Figures 5, a, b show the magnetization characteristics for the polar coordinate system \((H, \alpha)\), which are obtained by recalculating characteristics presented in Fig. 3, a, b. As in Fig. 3, the symbols « » mark the nodes of the «longitudinal» magnetization characteristics with known magnetic flux density values at \(H = 0:200:1200\) A/m, \(\alpha = 0: \pi / 12: \pi / 2\), and the symbols « » correspond to the calculated values obtained using the expression (15) for nodes of the «transverse» magnetization characteristics \(B_\perp (H, \alpha)\).

The validity of the obtained results is confirmed by numerical calculations. For previously accepted values of the magnetic field strength vector \(H^{(1)} = \begin{pmatrix} 450, 600 \end{pmatrix}\) A/m = \((H, \alpha) = (750 A/m, 39.06 ^\circ)\) and \(H^{(2)} = \begin{pmatrix} 1120, 375 \end{pmatrix}\) A/m = \((1181 A/m, 8.758 ^\circ)\) calculated values of the corresponding magnetic flux density vectors \(B^{(1)} = \begin{pmatrix} 1.574, 0.392 \end{pmatrix}\) T and \(B^{(2)} = \begin{pmatrix} 1.666, 0.283 \end{pmatrix}\) T. Exact values of magnetic flux density vectors obtained using analytical formulas from [8]: \(B^{(1)*} = \begin{pmatrix} 1.573, 0.394 \end{pmatrix}\) T, \(B^{(2)*} = \begin{pmatrix} 1.666, 0.286 \end{pmatrix}\) T.

In conclusion, we note that the integrand in the integral (1) characterizes the change in the density of the co-energy of the magnetic field spent on the cyclic magnetization of the medium. The results obtained can easily be transferred to a similar integral for the energy density \(HdB\), the use of which leads to the vector dependencies \(H(B)\), namely \(H_1(B_1, B_2)\) and \(H_2(B_1, B_2)\) or \(H_1(B, \alpha)\) and \(H_2(B, \alpha)\) depending on the selected coordinate system.

Examples of the use of the results obtained in relation to anisotropic electrical steels will be the subject of special consideration.

Conclusions.

1. The task of constructing the correct vector material equations for nonlinear anisotropic soft magnetic materials remains one of the main reserves for increasing the accuracy of mathematical models in solving magnetostatic problems in the field formulation.

2. An effective direction for solving this problem, which has been actively developing in recent years, is to use the energy approach to constructing the vector characteristics of magnetization. However, the impossibility of directly measuring energy potentials, the complexity of the analytical description and ensuring accuracy with double differentiation to determine the differential magnetic permeability tensor make the task of constructing them quite time-consuming.

3. An alternative technique of constructing the vector characteristics of magnetization is to use the reciprocity principle, which is valid for media with reversible magnetization processes. Its main advantage is the ability to directly recalculate the magnetization characteristics in one of the directions according to the specified magnetization characteristics in the orthogonal direction without calculating the energy potential.
4. The asymptotic expressions for the reciprocity principle established in this paper, which are universal in character for arbitrary magnetic media in the hysteresis-free approximation, open up additional possibilities for optimizing computational processes and increasing the accuracy of numerical methods for solving magnetostatic problems in the field formulation.

REFERENCES


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