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TESTING OF NUMERICAL SOLUTION OF THE PROBLEM OF DETERMINING SOURCES OF MAGNETOSTATIC FIELD IN MAGNETIZED MEDIUM

Purpose. Testing of numerical solution algorithm for integral equation for calculation of plane meridian magnetostatic field source distribution at interfaces of piecewise homogeneous magnetized medium by means of electrostatic analogy. Methodology. The piecewise homogeneous medium consists of three regions with different magnetic permeabilities: the shell of arbitrary meridian section, external unlimited medium outside the shell, and the medium inside the shell. For testing external homogeneous magnetic field effect on spherical shell is considered. The analytical solution of this problem on the basis of electrostatic analogy from the solution of the problem uniform electrostatic field effect on dielectric shell is obtained. We have compared results of numerical solution of integral equation with the data obtained by means of analytical solution at the variation of magnetic permeabilities of regions of medium. Results. Integral equation and the algorithm of its numerical solution for calculation of source field distribution at the boundaries of piecewise homogeneous medium is validated. Testing of integral equations correctness for calculation of fictitious magnetic charges distribution on axisymmetric boundaries of piecewise homogeneous magnetized medium and algorithms of their numerical solutions can be carried out by means of analytical solutions of problems of homogeneous electrostatic field effect analysis on piecewise homogeneous dielectric medium with central symmetry of boundaries – single-layer and multilayer spherical shells. In the case of spherical shell in wide range of values of the parameter λ_k, including close to ± 1, numerical solution of integral equation is stable, and relative error in calculating of fictitious magnetic charges surface density and magnetic field intensity inside the shell is from tenths of a percent up to several percent except for the cases of very small values of these quantities. Originality. The use analytical solutions for problems of calculation of external electrostatic field effect on piecewise homogeneous dielectric bodies for testing integral equations of magnetostatics and algorithms for their numerical solutions. Practical value. The described method of testing integral equations of magnetostatics and their numerical solutions can be used for calculation of magnetic fields of spacecraft control system electromagnets. References 12, tables 2, figures 3.

Key words: **plane meridian magnetostatic field, piecewise homogeneous magnetized medium, integral equation, electrostatic analogy, fictitious magnetic charge.**

Выполнена проверка правильности интегрального уравнения второго рода для расчета распределения источников плоскомеридианного магнитостатического поля на границах раздела кусочно-однородной намагничиваемой среды и его численного решения. Для этого использованы электростатическая аналогия и аналитическое решение задачи о воздействии однородного электростатического поля на сферическую диэлектрическую оболочку в кусочно-однородной диэлектрической среде. Подтверждена правильность интегрального уравнения и его численного решения при помощи аппроксимирующей системы алгебраических уравнений. Сделан анализ влияния магнитных проницаемостей однородных областей среды на распределение фиктивных магнитных зарядов на поверхностях и напряженность магнитного поля внутри сферической оболочки. Библ. 12, табл. 2, рис. 3.

Ключевые слова: **плоскомеридианное магнитостатическое поле, кусочно-однородная намагничиваемая среда, интегральное уравнение, электростатическая аналогия, фиктивный магнитный заряд.**

Introduction. For the calculation of magnetostatic fields in inhomogeneous magnetized media, the use of integral equations of the second kind with respect to the density of fictitious magnetic charges in the volume and on the interfaces of the sections of the medium is effective [1-3]. Integral equations are approximated on a spatial mesh by systems of algebraic equations of high order, which are solved using computers. As in the formulation of integral equations, and with their approximation, errors can be made, connected, for example, with inconsistencies in the directions of vectors, integrating on the elementary part of the computational domain with the singular point of the kernel of the integral equation.

The relevance of this work is due to the need to verify the correctness of the used algorithms and labor-intensive computation procedures using tasks that have analytical (exact) solutions – testing. The number of such solutions in magnetostatics is relatively small. In the known papers, exact solutions of problems of calculating analogous physical fields are not fully utilized, giving preference to more accurate, in the opinion of the authors, numerical methods.

The goal of the work is use of electrostatic analogy for testing the algorithm for the numerical solution of the integral equation for the surface density of fictitious magnetic charges at the interfaces of homogeneous regions of a piecewise homogeneous magnetized medium in the case of a plane meridian magnetostatic field.

Main equations and formulae. Let it be required to test the algorithm for solving a problem for a piecewise homogeneous medium consisting of three homogeneous regions with different constant absolute magnetic permeabilities μ_k ($k = \overline{1,3}$). The shell of an arbitrary meridian section (region 2) divides the unbounded environment into regions 1 and 3, respectively, outside and inside the shell (Fig. 1). In the particular case, region 3 is absent, i.e. there is an axisymmetric body in an unbounded medium, for example, the core of an electromagnet. Using the electrostatic analogy of the problem under consideration [1, 4-6], we represent the scalar potential φ_m of the magnetostatic field due to the magnetic properties of the medium in the form [3, 7, 8]:

$$\varphi_m(Q) = \frac{1}{\pi\mu_0} \int_l \frac{\sigma_m(M) r_M K(k)}{\sqrt{(z_Q - z_M)^2 + (r_Q + r_M)^2}} dl_M , \quad (1)$$

where Q, $M \in l$ are the point of observation and point with current coordinates, respectively; μ_0 is the magnetic constant; $\sigma_m(M)$ is the surface density of fictitious magnetic charges; l, dl_M are the total contour of the meridian section of the shell and its element with center at the point M, respectively; $l = l_1 + l_2$, $l_{1,2}$ are the outer and inner parts of the total contour, respectively; $K(k)$ is the complete elliptic integral of the first kind of module k [9];

$$k = 2\sqrt{\frac{r_Q r_M}{(z_Q - z_M)^2 + (r_Q + r_M)^2}};$$

r_Q, r_M and z_Q, z_M are the radial and axial cylindrical coordinates of points Q and M.

The strength of the magnetic field due to the magnetic properties of the medium, and the resulting magnetic field are respectively equal [1]

$$\vec{H}_m = -\operatorname{grad}\varphi_m \qquad (2)$$

and

$$\vec{H} = \vec{H}_0 + \vec{H}_m, \qquad (3)$$

where $\vec{H}_0$ is the external magnetic field strength.

Following the idea of the method [1], we note that in order to perform calculations using formulae (1) – (3), it is necessary to find an unknown function $\sigma_m(Q)$, $Q \in l$ by solution of the integral equation

$$\sigma_m(Q) - \frac{\lambda_k}{\pi}\int_l \sigma_m(M) S(Q,M)\, dl_M = 2\mu_0 \lambda_k H_{0n}(Q), \qquad (4)$$

where

$$S(Q,M) = \frac{k}{2\sqrt{r_Q^3}}\left\{\sqrt{r_M}\left[K(k) + \frac{1}{k'^2}\times\right.\right.$$
$$\left.\times\left(\frac{r_M + r_Q}{2r_M}k^2 - 1\right)E(k)\right]\cos(\vec{1}_r, \vec{n}_Q) + \qquad (5)$$
$$\left.+\frac{z_Q - z_M}{2\sqrt{r_M}}\left(\frac{k}{k'}\right)^2 E(k)\cos(\vec{1}_z, \vec{n}_Q)\right\};$$

$\vec{1}_r$, $\vec{1}_z$ are the orts of cylindrical coordinates r and z; $\vec{n}_Q$ is the unit normal to the contour l in the point $Q \in l$; $E(k)$, k' is the complete elliptic integral of the second kind of the module k and the additional module of complete elliptic integrals [9]; $k' = \sqrt{1 - k^2}$;

$$\lambda_k = \frac{\mu_{k+1} - \mu_k}{\mu_{k+1} + \mu_k}, \; k = 1, 2;$$

$H_{0n}(Q)$ is the normal projection of $\vec{H}_0$.

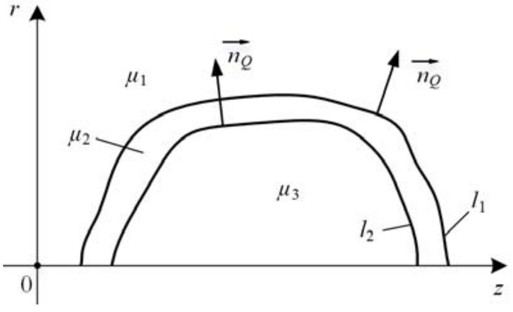

Fig. 1. An axisymmetric shell in a piecewise homogeneous magnetized medium

The particular case of the problem under consideration. For testing, we consider a particular case of the problem described above – the effect of external constant homogeneous magnetic field directed along the axial coordinate z on the spherical shell in a piecewise homogeneous magnetized medium (Fig. 2). The meridian section of this shell is symmetric about the r axis, therefore, for the points M and M' with such symmetry $\sigma_m(M') = -\sigma_m(M)$ and the domain of definition of $\sigma_m(M)$ is halved. We transform the integral equation (4) for this case to the form:

$$\sigma_m(Q) - \frac{\lambda_k}{\pi}\int_l \sigma_m(M)\left[S(Q,M) - S(Q,M')\right] dl_M = \qquad (6)$$
$$= 2\mu_0 \lambda_k H_0 \sin\theta,$$

where θ is the spherical coordinate of elevation angle (Fig. 2).

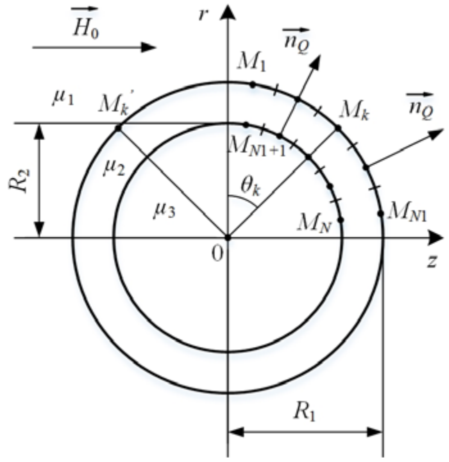

Fig. 2. A spherical shell in a piecewise homogeneous magnetized medium

The total integration contour l in equation (6) consists of two halves l_1 and l_2 symmetric about the r-axis, located in the region $z \geq 0$. The functions $S(Q, M)$ и $S(Q, M')$ entering the kernel of this equation, we determine by formula (5), having adopted in it $\cos(\vec{1}_r, \vec{n}_Q) = \cos\theta$, $\cos(\vec{1}_z, \vec{n}_Q) = \sin\theta$, $z_{M'} = -z_M$, $r_{M'} = r_M$. In addition, it is necessary to take into account the change in the z-coordinates of the symmetric points M' in calculating the modulus k. After solving equation (6), the intensity of the homogeneous magnetic field at an arbitrary point Q inside the shell (region 3) is found using the formula that follows from (1) – (3):

$$H_i(Q) = H_0 + \frac{1}{8\pi\mu_0}\int_l \sigma_m(M)\times$$
$$\times\frac{1}{r_Q\sqrt{r_Q r_M}}(S_1 - S_1')\, dl_M, \qquad (7)$$

where

$$S_1 = (z_Q - z_M)\frac{k^3}{k'^2}E(k). \qquad (8)$$

The function S_1' in the integrand of the second term (7) is determined by formula (8), replacing in it, as well as in the formula for determining k, the coordinate z_M on $z_{M'}$.

When forming the functions $S(Q, M)$ and $S(Q, M')$ entering the kernels of equations (4), (6), formulas were used to calculate the projections of the plane meridian electrostatic field from [3, 8]. The contour l was divided into N elementary regions with nodal points M_k at their center, which form a spatial mesh, and $k=\overline{1,N_1}$ at $M_k \in l_1$ and $k=\overline{N_1+1,N}$ at $M_k \in l_2$ (Fig. 2). Equation (6) was transformed into a system of algebraic equations on a mesh using the quadrature formula of rectangles. The diagonal elements of the $N\times N$ matrix of this system of equations, corresponding to the elementary sections of the contour with the singular point of the kernel of equation (6), were determined by the method described in [10]. The system of algebraic equations was solved by a direct method based on the inversion of the matrix of the left-hand sides and the subsequent multiplication of the inverse matrix by the column vector of the right parts.

Analytical solution of a similar electrostatic problem of the action of an external homogeneous electric field on a dielectric spherical shell is known [11]. Using this solution, we obtain formulas for calculating the distribution of the surface density of fictitious magnetic charges on the boundary surfaces, as well as the intensity of the homogeneous magnetic field H_i inside the magnetized shell:

$$\sigma_m(R_1,\theta)=\mu_0\left(\frac{2}{R_1^3}(B_1-B_2)+H_0+A_2\right)\sin\theta\,; \quad (9)$$

$$\sigma_m(R_2,\theta)=\mu_0\left(\frac{2B_2}{R_2^3}+H_i-A_2\right)\sin\theta\,; \quad (10)$$

$$H_i=-9H_0\Bigg/\left[c_{\mu 1}\left(\frac{\mu_2}{\mu_1}+2\right)-2c_{\mu 2}\left(\frac{R_2}{R_1}\right)^3\left(\frac{\mu_2}{\mu_1}-1\right)\right], \quad (11)$$

where R_1, R_2 are the radii of the boundary surfaces (Fig. 2);

$$B_1=R_1^3\left[H_0+A_2\left(1+c_\mu\left(\frac{R_2}{R_1}\right)^3\right)\right];\; B_2=c_\mu R_2^3 A_2;$$

$$c_{\mu 1}=\mu_3/\mu_2+2\,;\; c_{\mu 2}=1-\mu_3/\mu_2\,;\; c_\mu=c_{\mu 2}/c_{\mu 1}\,;$$

$$A_2=H_i c_{\mu 1}/3\,.$$

The values of σ_m and H_i obtained by numerical solution of the integral equation (6) and calculations by the formula (7) will be called approximate, and using (9) – (11) – exact.

Table 1, 2 show the values of $\sigma_m^* = \sigma_m / (\mu_0 H_0)$ and $H_i^* = H_i / H_0$, respectively, and Fig. 3 shows the variation curves for σ_m^* vs θ [0, π/2] on the boundary surfaces of the shell at $\mu_1 = \mu_0$, $R_2/R_1 = 0.95$ and the variation of $\mu_{2,3}$. The data in columns 1 are approximate, and in columns 2 – exact. For the data given in the numerators of columns 1 of Table 1, it was assumed that N = 80, in the denominators – 2160. The curves in Fig. 3 are built from the results of a numerical solution of equation (6) with N = 2160.

From Table 1, 2 it follows that in wide ranges of variation of the magnetic permeabilities μ_2 и μ_3 when the step of the spatial mesh is reduced the absolute discrepancies of the exact and approximate values of σ_m^* и H_i^* are of the order of 10^{-3}. In this case, the relative discrepancies vary from 0.1% to several percent, except for very small values of the calculated value.

Table 1

The values of the surface density of fictitious magnetic charges σ_m^* on the surfaces of a spherical shell

The shell surface	θ_k, рад	$\mu_2 = 50\mu_0$, $\lambda_1 = 0.961$						$\mu_3 = \mu_0$					
		$\mu_3 = 10\mu_0$, $\lambda_2 = -0.667$		$\mu_3 = 100\mu_0$, $\lambda_2 = 0.333$		$\mu_3 = 1000\mu_0$, $\lambda_2 = 0.905$		$\mu_2 = 50\mu_0$, $\lambda_1 = 0.961$, $\lambda_2 = -0.961$		$\mu_2 = 500\mu_0$, $\lambda_1 = 0.996$, $\lambda_2 = -0.996$		$\mu_2 = 2000\mu_0$, $\lambda_1 = 0.999$, $\lambda_2 = -0.999$	
		1	2	1	2	1	2	1	2	1	2	1	2
Outer	0.2945	0.7312 / 0.7483	0.7490	0.8288 / 0.8348	0.8350	0.8457 / 0.8497	0.8498	0.6067 / 0.6370	0.6383	0.7823 / 0.8339	0.8362	0.8048 / 0.8594	0.8617
	0.6086	1.4404 / 1.4739	1.4753	1.6326 / 1.6443	1.6447	1.6658 / 1.6736	1.6739	1.1951 / 1.2548	1.2573	1.5412 / 1.6426	1.6470	1.5855 / 1.6928	1.6974
	0.9228	2.0087 / 2.0553	2.0572	2.2765 / 2.2929	2.2935	2.3229 / 2.3338	2.3342	1.6668 / 1.7497	1.7532	2.1499 / 2.2906	2.2967	2.2117 / 2.3605	2.3670
	1.2369	2.3812 / 2.4355	2.4378	2.6977 / 2.7171	2.7178	2.7525 / 2.7656	2.7660	1.9765 / 2.0735	2.0776	2.5502 / 2.7144	2.7216	2.6236 / 2.7972	2.8049
	1.5510	2.5335 / 2.5774	2.5798	2.8539 / 2.8754	2.8762	2.9033 / 2.9267	2.9272	2.1023 / 2.1943	2.1986	2.7249 / 2.8725	2.8802	2.8047 / 2.9602	2.9683
Inner	0.2945	–0.0507 / –0.0446	–0.0443	0.0103 / 0.0090	0.0089	0.0208 / 0.0182	0.0181	–0.1286 / –0.1134	–0.1128	–0.0441 / –0.0184	–0.0173	–0.0329 / –0.0057	–0.0045
	0.6086	–0.0999 / –0.0878	–0.0872	0.0203 / 0.0177	0.0176	0.0410 / 0.0359	0.0357	–0.2533 / –0.2234	–0.2222	–0.0868 / –0.0362	–0.0340	–0.0647 / –0.0112	–0.0089
	0.9228	–0.1393 / –0.1224	–0.1217	0.0282 / 0.0247	0.0246	0.0572 / 0.0501	0.0497	–0.3530 / –0.3116	–0.3098	–0.1206 / –0.0505	–0.0474	–0.0898 / –0.0156	–0.0124
	1.2369	–0.1646 / –0.1450	–0.1442	0.0334 / 0.0293	0.0291	0.0678 / 0.0593	0.0590	–0.4174 / –0.3692	–0.3671	–0.1410 / –0.0598	–0.0562	–0.1044 / –0.0185	–0.0147
	1.5510	–0.1319 / –0.1534	–0.1526	0.0139 / 0.0310	0.0308	0.0150 / 0.0627	0.0624	–0.3902 / –0.3906	–0.3885	–0.0764 / –0.0632	–0.0595	–0.0349 / –0.0195	–0.0155

Table 2
The values of the magnetic field intensity H_i^* penetrated inside the spherical shell

N	$\mu_2 = 50\mu_0$, $\lambda_1 = 0.961$				$\mu_3 = \mu_0$			
	$\mu_3 = 10\mu_0$, $\lambda_2 = -0.667$		$\mu_3 = 100\mu_0$, $\lambda_2 = 0.333$		$\mu_2 = 50\mu_0$, $\lambda_1 = 0.961$, $\lambda_2 = -0.961$		$\mu_2 = 500\mu_0$, $\lambda_1 = 0.996$, $\lambda_2 = -0.996$	
	1	2	1	2	1	2	1	2
80	0.2179	0.1908	0.0363	0.0308	0.4502	0.3965	0.1507	0.0596
240	0.2006		0.0325		0.4159		0.0935	
720	0.1941		0.0314		0.4031		0.0711	
2160	0.1919		0.0310		0.3987		0.0634	

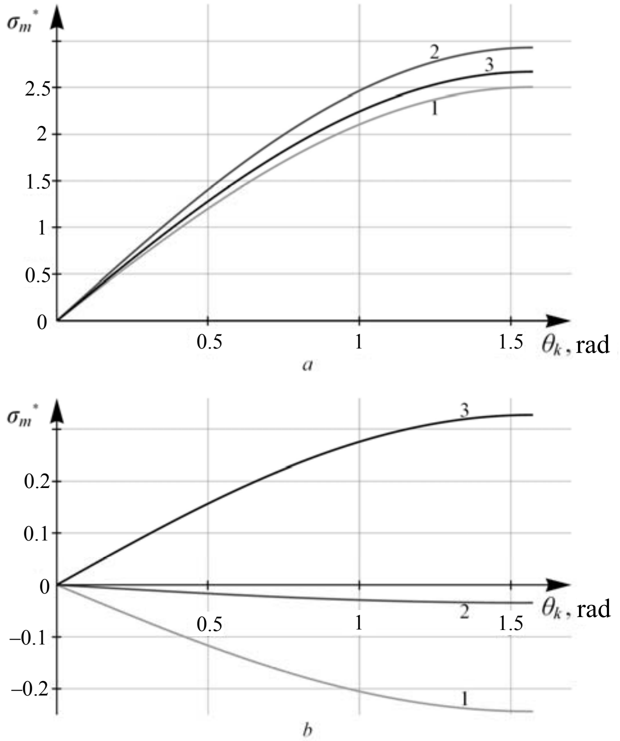

Fig. 3. The distributions of σ_m^* on the outer (a) and inner (b) surfaces of the spherical shell: for curve 1 $\mu_2 = 100\mu_0$, $\mu_3 = \mu_0$, 2 – $\mu_2 = 1000\mu_0$, $\mu_3 = \mu_0$, 3 – $\mu_2 = 10\mu_0$, $\mu_3 = 5000\mu_0$

Increasing the accuracy of the numerical solution can also be achieved by using more precise quadrature formulas. From Table 1 and Fig. 3 it follows that the numerical solutions are stable for values of the parameter λ_k close to ± 1. Thus, with the help of an analytical solution of a similar electrostatic problem, the correctness of the compilation of the original integral equation (4) and the algorithm for its numerical solution was confirmed. We note that analytical solutions of problems of the effect of external homogeneous electrostatic field on multilayer dielectric spherical shells are known [12], which can be used to test the algorithm for solving the integral equation in the case of multilayer axisymmetric magnetized shells.

The features of the variation of σ_m^* and H_i^* with the variation of μ_k in the case of a spherical shell. The values of σ_m^* on the calculated part of the contour of the outer surface of the spherical shell ($z > 0$) are positive for all μ_k and vary insignificantly for large μ_2. On the calculated part of the contour of the inner surface of the shell for $\mu_3 < \mu_2$ ($\lambda_2 < 0$), the values of σ_m^* are negative, and for $\mu_3 > \mu_2$ ($\lambda_2 > 0$) are positive (Table 1). Naturally, on the part of the contour symmetric about the r-axis, the signs of σ_m^* are opposite.

At large $\mu_2 \geq 100\mu_0$, the shell shields region 3, which results in small values of H_i^* and σ_m^*. An increase in μ_3 leads to an additional decrease in H_i^* (Table 2).

The described features of the changes of σ_m^* and H_i^* can also be useful in the analysis of the magnetostatic field in the case of axisymmetric shells and solid bodies of a different shape.

Conclusions.

1. Testing the correctness of integral equations for calculating the distribution of fictitious magnetic charges on the axisymmetric boundaries of a piecewise homogeneous magnetized medium and the algorithms of their numerical solutions can be carried out using analytical solutions of problems of analyzing the action of homogeneous electrostatic field on a piecewise homogeneous dielectric medium with a central symmetry of boundaries – single-layer and multilayer spherical shells.
2. In the case of a spherical shell in a wide range of values of the parameter λ_k including those close to ± 1, the numerical solution of the integral equation is stable, and when reducing the mesh step, the relative error in calculating the surface density of fictitious magnetic charges and the magnetic field strength inside the shell is from the tenths percentages to a few percent, except for very small values of these values.

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